Upon substitution of (40) into the governing equation (23), we see that (23) will be satisfied identically provided

$$\sum_{j=1}^{\infty} \frac{C_j}{A_j} J_0(\lambda_j r) \left(\frac{1}{\theta_1} - \upsilon\lambda_j^2\right) = \frac{P_g}{\rho L} \tag{41}$$

If we prescribe that

$$A_j = \left(\frac{1}{\theta_1} - \upsilon\lambda_j^2\right) \tag{42}$$

then (41) becomes

$$\sum_{j=1}^{\infty} C_j J_0(\lambda_j r) = \frac{P_g}{\rho L} \tag{43}$$

According to the Fourier-Bessel expansion, the constants C_j are given as

$$C_j = \frac{\frac{P_g}{\rho L} \int_0^{R_0} r\, J_0(\lambda_j r)\, dr}{\frac{1}{2} R_0^2 \left[J_0^2(\lambda_j R_0) + J_1^2(\lambda_j R_0) \right]} \tag{44}$$

From equation (40) we see that the boundary condition (26) will be satisfied providing we define λ_j as the positive roots of the equation

$$J_0(\lambda_j R_0) = 0 \tag{45}$$

Combining equation (44) and (45), we have

$$C_j = \frac{2P_g}{\rho L R_0 \lambda_j} \left[\frac{1}{J_1(\lambda_j R_0)}\right] \tag{46}$$

Inserting (42) and (46) into (40), we obtain

$$w(r,t) = \frac{2P_g}{\rho L R_o} \sum_{j=1}^{\infty} \frac{J_o(\lambda_j r)}{\lambda_j J_1(\lambda_j R_o)} \left[\frac{1}{\frac{1}{\theta_1} - \upsilon\lambda_j^2}\right] \cdot \left[e^{-\lambda_j^2 \upsilon t} - e^{-t/\theta_1}\right] \tag{47}$$

An inspection of equation (47) shows that conditions (24), (25), and (26) are satisfied. Thus equation (47) is the formal solution of the subject boundary value problem.

The volume rate of fluid Q passing through the tube in time t can be determined as follows:

$$Q = \int_o^A w(r,t)\, da \tag{48}$$

where A is the cross-sectional area of the bore of the knock-off tube. Substituting equation (47), we have

$$Q = \frac{4\pi P_g}{\rho L R_o} \sum_{j=1}^{\infty} \left[\frac{1}{\frac{1}{\theta_1} - \upsilon\lambda_j^2}\right] \left[e^{-\lambda_j^2 \upsilon t} - e^{-t/\theta_1}\right] \cdot \left[\frac{1}{\lambda_j J_1(\lambda_j R_o)}\right] \int_o^{R_o} r J_o(\lambda_j r)\, dr \tag{49}$$

from which

$$Q = \frac{4\pi P_g}{\rho L} \sum_{j=1}^{\infty} \left[\frac{1}{\frac{1}{\theta_1} - \upsilon\lambda_j^2}\right] \left[e^{-\lambda_j^2 \upsilon t} - e^{-t/\theta_1}\right] \left(\frac{1}{\lambda_j^2}\right) \tag{50}$$